Approximating Correlated Defaults for Credit Default Options and Swaps

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Summary

I propose statistical approximations to model correlated defaults.

The approximations follow from a structural risk factor approach and can price credit default options as well as credit default swaps with an assumed loss given default.

The approach also yields metrics characterizing portfolio default risk and default-relative diversification.

Why Default Correlation Matters

Default correlation modeling is crucial to pricing bond portfolios and derivatives.

These portfolios arise naturally in bank holdings; portfolio tranches arise from collateralize debt obligations. Portfolios and tranches underlie credit default options and swaps.

Structuring can strengthen the correlations in tranches — making them far more risky than apparent.

Ignoring or improperly modeling these correlations has nearly bankrupted many financial institutions recently.

Similar Work

Times to default are modeled using Erlang's (1909) assumption that a single delay is exponentially distributed.

Default Times

Jarrow and Turnbull (1995):
- first modeled default times as exponential.

Jarrow, Lando, and Turnbull (1997):
- allowed rates to differ by credit rating; and,
- allowed for Markov rating switches.

Banasik, Crook, and Thomas (1999):
- considered exponential and Weibull default times.

Collin-Dufresne, Goldstein, and Hugonnier (2004):
- explored random rate exponential default times; and,
- applied this to a two-bond CDO.

Correlated Default Times

Jarrow and Yu (2001):
- studied default by issuers with cross-holdings; but,
- complexity limited their study to two bonds.

Duffie, et al. (2009):
- used (similar) frailty-correlated default; however,
- purely data-driven; no asymptotic approximation.

Small-scale Asymptotics

Thiele (1871), Gram (1883), and Edgeworth (1883):
- theory to approximate distributions;
- approximations known as Edgeworth expansions.

McCullagh (1987):
- best explains non-standard Edgeworth expansions.

Cox and Barndorff-Nielsen (1989):
- hinted at approximations similar to those here.

Example: 200-bond CDO

Tranche	# Loans	Percent
A	150	75%
Mezzanine	40	20%
Equity	10	5%.

Correlations induced by a rare one-time shock.
Post-shock, loans default at δ times their normal rate.

Finally, we make some modeling assumptions:

1. default rates covering average (unshocked) default times of 5–20 years, skewed toward lower default rates:
$$\{\lambda\}_i = \left\{ \frac{10^{i/200}}{20} : i \in 1, \ldots, 200 \right\};$$
2. a systematic shock rate of $\lambda_s = 0.05$ (mean time-to-shock of 20 years) inducing correlations*; and,
3. the post-shock default rate acceleration $\delta = 5$.

This models a portfolio of B–C-rated loans with default rates like C–D-rated loans in the event of a severe recession.

* pre-reaction correlations from 0.048 (λ_i's near 0.5) to 0.330 (λ_i's near 0.05).

Edgeworth Expansions for Delays

We match base distribution parameters to the mean and variance of default times.

Standard Normal-based Expansion

$$\hat{f}_{\bar{Y}}(y) = \frac{\phi(z)}{\sqrt{\kappa_2}} \left[1 + \frac{\kappa_3(z^3 - 3z)}{6\sqrt{\kappa_2^3}} + \frac{\kappa_4(z^4 - 6z^2 + 3)}{24\kappa_2^2} + \frac{\kappa_3^2(z^6 - 15z^4 + 45z^2 - 15)}{72\kappa_2^3} \right] + O(n^{-3/2})$$

where $z = (y - \kappa_1)/\sqrt{\kappa_2}$ and $\phi(z)$ is the standard normal pdf.

Derived Gamma-based Expansion

$$\begin{aligned}\hat{f}_{\bar{Y}}(y) = \gamma_{\hat{m},\hat{\lambda}}(y) &+ \frac{\kappa_3\hat{\lambda}^3 - 2\hat{m}}{6} \sum_{j=0}^{3} (-1)^{3-j} \binom{3}{j} \gamma_{\hat{m}-j,\hat{\lambda}}(y) \\ &+ \frac{\kappa_4\hat{\lambda}^4 - 6\hat{m}}{24} \sum_{j=0}^{4} (-1)^{4-j} \binom{4}{j} \gamma_{\hat{m}-j,\hat{\lambda}}(y) \\ &+ \frac{(\kappa_3\hat{\lambda}^3 - 2\hat{m})^2}{72} \sum_{j=0}^{6} (-1)^{6-j} \binom{6}{j} \gamma_{\hat{m}-j,\hat{\lambda}}(y) + O(n^{-3/2}),\end{aligned}$$

where $\hat{m} = \kappa_1^2/\kappa_2$, $\hat{\lambda} = \kappa_2/\kappa_1$, and $\gamma_{m,\lambda}(y) = \text{Gamma}(m, \lambda)$ pdf if $m > 0$, else 0.

The form, binomial sums of other gamma densities, is both elegant and, to my knowledge, a new result.

Mélange Gamma base with normal-derived correction terms.
Log-Densities Log-densities approximations guarantee positivity.

Simulation: 200-Bond CDO Equity Tranche

Average tranche default time: sample mean ($\bar{Y}$) of ten smallest random variables. Simulated $\bar{Y}$'s cumulants and implied parameters:

$\hat{\kappa}_1$	$\hat{\kappa}_2$	$\hat{\kappa}_3$	$\hat{\kappa}_4$	$\hat{\lambda}$	$\hat{m}$
0.142	2.583×10^{-3}	1.018×10^{-4}	6.458×10^{-6}	55.12	7.849

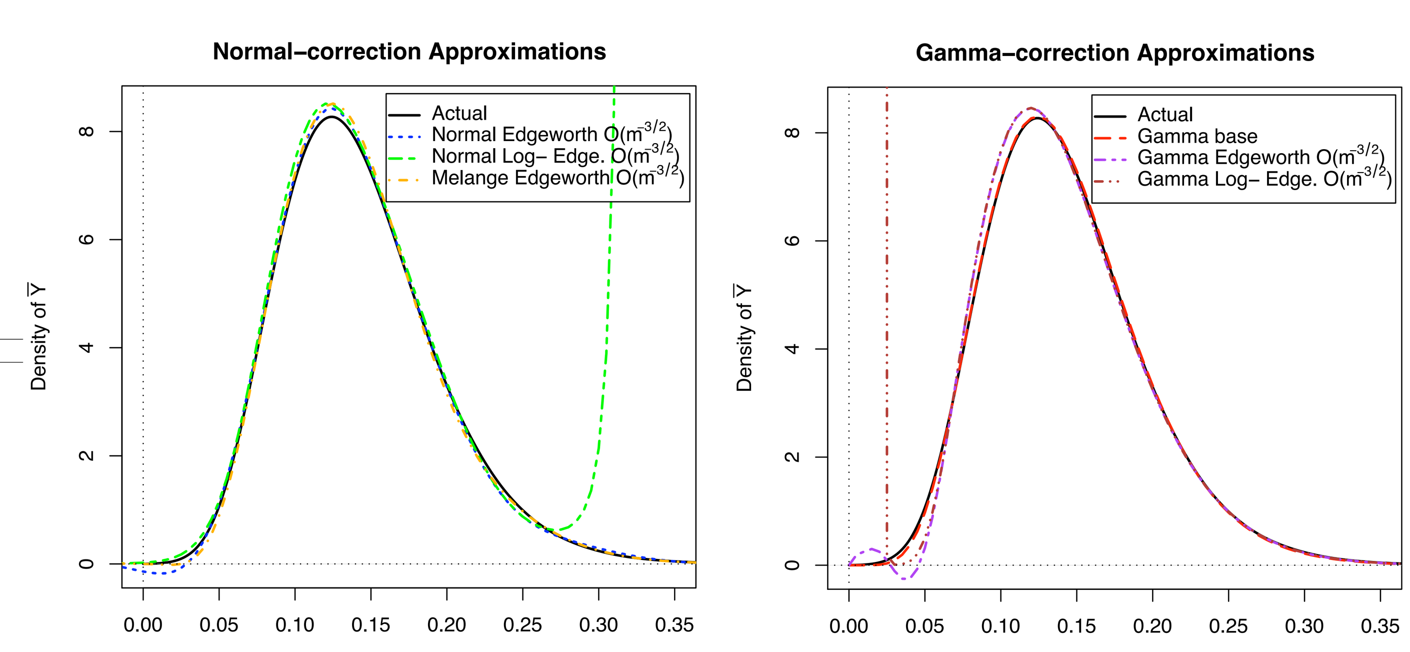
Average simulated default time: ≈ 2 months.

New Diversification Measure $\hat{m}$ is the number of iid loans needed to replicate the average default time distribution. This iid-equivalent loan count (ILC) is a measure of default-relative portfolio diversification.

The simulated CDO equity tranche has an ILC of $\hat{m} = 7.8$. Correlations and structuring reduce default-relative diversification by 22%.

Density Approximation Plots

Average default time density versus approximations:



- Normal approximation: minor negativity for $\bar{y} < 0.04$.
- Mélange expansion: almost no negativity; often almost identical to the actual density.
- Gamma base: almost identical to the actual density.
- Gamma approximation: positive for $0 < \bar{y} < 0.05$).
- Log-density expansions: very good in center; explode in tails.

Density Approximation Errors

Mean squared errors for approximations:

Approximation	Normal Edgeworth	Mélange Edgeworth	Gamma Base	Gamma Edgeworth
Mean Squared Error	0.00032	0.00013	0.00018	0.00154

The "wiggling" of the gamma-based Edgeworth expansion clearly increases the approximation mean squared error.

Notation

k = the number of risk factors;
m = the number of loans and risk factors;
X_i = default time of loan $i \in \{1, \ldots, m\}$;
λ_i = the rate parameter characterizing default time X_i; and,
$\bar{Y}$ = the average time to loan default.

Homogeneous Borrowers

If m borrowers all default at the same exponential rate λ, the average default time is Gamma(m, λ) distributed.

Borrowers of differing-size loans alter the average default time distribution. However:
Theorem 1 If default rates scale with loan size, then the average default time is still Gamma(m, λ) distributed.

Heterogeneous Borrowers/ Common Risk Factors

We might know nothing about the homogeneity of borrowers. Worse, shared risk factors might make defaults correlated between borrowers.

Delay Form Average delays must be weighted sums of sub-delays. Weighting is equivalent to scaling the rate parameter λ. Thus average delay: $Y = \sum_{i=1}^{m+k} X_i$ where $X_i \sim \text{Exp}(\lambda_i)$ for all i.

Sub-Delay Partition We must be able to partition sub-delays into non-empty idiosyncratic and systematic (shared) terms.

Consistency **Theorem 2** Given the above, Edgeworth expansions are consistent for the average delay distribution. (Thus Edgeworth expansions will work for all our cases.)

From Average Default Times to Complete Default Times

The approximate average default density $\hat{f}_{\bar{Y}}$ can yield insights into complete portfolio default CDF F.

Three ways to infer the complete default CDF F:

1. Take gamma base as truth;
   - Assume portfolio holds $\hat{m}$ iid Exp($\hat{\lambda}$) loans;
   - Complete default CDF : $\hat{F}(y) = (1 - e^{-\hat{\lambda}y})^{\hat{m}}$.
2. Get CDF from gamma-based expansion;
   - Integrate gamma-based expansion $\hat{f}_{\bar{Y}}$ to get $\hat{F}_{\bar{Y}}$;
   - Complete default CDF: $\hat{F}(y) = [\hat{F}_{\bar{Y}}(y)]^{\hat{m}}$.
3. Use expansions with complete default cumulants.
   - Predict complete default κ's via survival analysis;
   - Preceding expansions are still valid.

The third method seems the most promising: Inference without survival analysis would probably yield biased default time predictions.

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