Approximating Correlated Defaults

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Introduction

- In the 2008–2009 financial crisis:
  - US households alone lost $11 Tn in wealth; and,
  - Structured debt products had impairments of over $1.5 Tn.
- Key stylized fact: accelerated, clustered defaults on loans.
- Defaults affect portfolios of loans like those held by banks.
- Defaults central to structured debt (*i.e.* portfolio) products.
  - Allocate risks via securitization to lower borrowing costs.
  - CMOs (prepayment risk[1]); CDOs, CDSs (default risk).

[1] Defaults may cause prepayments on loans backed by a guarantor, *e.g.* FNMA.

Portfolio Default Risk

- Must measure portfolio risk; cannot assume independence.
  - Typical portfolio metric: correlation/covariance matrix.
  - However, correlation is linear; default is non-linear.
- Ideally, we would like to:
  - Understand default dependence/clustering; and,
  - Measure portfolio diversification.
- Past approaches (copulas, Moody's KMV) clearly failed.

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Results Preview

- Consider intuitive default behavior (crisis acceleration).
- Current models (affine, exponential) cannot handle this.
  - The state-of-the-art has problems with such behavior.
  - May also explain need for seasoning period.
- Find approximation that is elegant, consistent, and novel.
- Yields default-approximating portfolio of iid bonds/loans.
- First theory for jointly determining two useful risk metrics:
  - # loans in approximating portfolio ("diversity score");
  - average default rate of those iid loans.
- Includes corrections to address possibly heavy tails.
- Finally: lets us approximate the default-time distribution.

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Why Not a Structural Model?

- Two approaches to defaults: structural and reduced-form.
- Structural: assets evolve randomly; default barrier.
  - Merton (1974), Black and Cox (1976), Leland and Toft (1996).
  - Zhou (2001) uses asset correlations for multi-firm model.
- However, there are problems with structural models.
  - Giesecke (2006): problems if assets not directly observed.
  - Worse: Very hard to get any default correlation measure.
- This is why most recent work uses reduced-form approach.
  - I will focus on a reduced-form (statistical) approach.

Structured Debt Products

- Collateralized mortgage obligations (CMOs) allocate prepays.[2]
- Collateralized debt obligations (CDOs) allocate defaults.
- Tranches set priority of who incurs defaults, prepays.
- Credit default swaps (CDSs), written on bonds/CDOs.
- CDOs and CDSs are just bond/bond portfolio derivatives.
- Difficulty of modeling CDOs/CDSs: they involve defaults.
    - Tough because defaults are rare events.
- CDO tranche (*e.g.* first 5% of defaults) seems harder.
- Turns out some theory handles tranches with ease.

[2] Prepayments are sometimes triggered by defaults.

Time to Default

- Think of defaults as time to default, loss given default.
  - Our concern here: Time to default $\cong$ PD, default rate.
- Often model waiting times as exponentially-distributed.
  - Like flipping a coin periodically: heads $=$ default occurs.
  - Erlang (1909) used this as distribution theory for delays.
- Thus the reduced-form approach: Model default rates (times).
- Exponentially-distributed default times:
  - Jarrow and Turnbull (1995), Jarrow *et al* (1997), Banasik *et al* (1999), Collin-Dufresne *et al* (2004)
- Are unconditionally-observed default times exponential? (No.)

Correlated Defaults: Why?

- However, it seems likely that defaults are related.
- More simply:
  - Do we think defaults stay constant in recessions?
  - Are laid-off coworkers all more likely to default?
- Formally: Borrowers may share certain risk factors.
  - Sensitivity to national, local economy; and,
  - Sensitivity to certain industries, companies.
- Past few years: record losses on portfolio defaults.
- Seems default correlations/dependence were not well-modeled.

Correlated Defaults: Difficult

- Why were these dependence structures ignored?
- Jarrow and Yu (2001) modeled 2 bonds with cross-holding issuers.
  - For more bonds "working out these distributions is more difficult."
- Even harder if we are considering a tranche of a CDO.
- What if we knew each bond's default distribution, dependences?
  - Exact portfolio default distribution is still very difficult.
- Problem: Still want metrics for effect of correlated defaults.

Correlated Defaults: Current Approaches

- How to model default correlations/dependence?
  - One way: Copulas. Easy to use — but opaque, nonlinear.
  - More recent focus: better modeling of default rates.
- Work on better models of default rates:
  - Duffie and Gârleanu (2001): systematic, idiosyncratic components.
  - Giesecke (2003): Marshall-Olkin default correlations. (!)
  - Duffie *et al* (2009): linear model of default intensity.
- I approximate average portfolio default distribution.

Affine versus Non-Affine Models

- Prior work has largely assumed affine models.
  - Default rate is linear function/model.
  - This assumption makes the math easier.
- Unfortunately, this has troubling implications:
  - Recession fixed effect $\Rightarrow$ AAA more affected than B, C.
- Instead, we explore a multiplicative in-crisis effect.
- Multiplicative effect $\Rightarrow$ exponential approaches incorrect.
  - Plainly: Current models cannot handle this behavior.

Systematic and Idiosyncratic Defaults

- First assume a simple reduced-form model:
  - Events occur when random "alarm clocks" go off[3].
  - Alarms related to systematic (common) risk factors; and,
  - Alarms related to idiosyncratic (borrower-specific) risks.
- The model dynamics can then be thought of as:
  - When idiosyncratic alarm rings, that borrower defaults.
  - When systematic alarm rings, macro event occurs (*e.g.* US recession).
  - Idiosyncratic clocks then speed up for exposed borrowers.
- This allows a statistical approximation (Edgeworth expansion).

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[3] The exponential timers we refer to in stochastic processes.

Edgeworth Expansions

- Edgeworth expansion: base distribution plus correction terms.
- Expansions use cumulants (like centered moments).
  - First four cumulants: mean, variance, skewness, kurtosis.
- Cumulants determine base distribution parameters, corrections.
- Typically, the base distribution is the normal distribution.
- Instead, I expand about a gamma distribution:
  - Sum of iid exponential random variables is gamma-distributed.
  - Sum of non-iid, correlated exponential r.v.s?
  - Base gamma distribution implied by cumulants is close.

Gamma Distribution vs. Exponential

- Before expanding about gamma, look at gamma pdf.
- Let Y be the average default time, then:

$$f_Y(y) = \gamma_{\ell,\lambda}(y) \tag{1}$$
$$= \frac{\lambda^\ell}{\Gamma(\ell)} y^{\ell-1} e^{-\lambda y} \tag{2}$$

- Mean, variance, skewness, kurtosis: $\frac{\ell}{\lambda}, \frac{\ell}{\lambda^2}, \frac{2\ell}{\lambda^3}, \frac{6\ell}{\lambda^4}$.
- Compare gamma to exponential distribution:

$$f_Y(y) = \lambda e^{-\lambda y}. \tag{3}$$

- Exponential: no seasoning; most defaults after issuance

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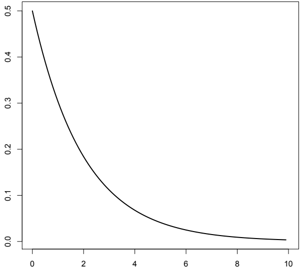
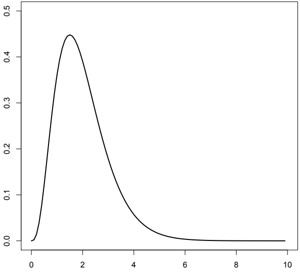
Gamma Distribution vs. Exponential: Plots



$\text{Exp}(\lambda = 0.5)$



$\text{Gamma}(\ell = 4, \lambda = 2)$

- Both have same mean time to default: two years.
- Current affine models use the exponential distribution (left).
- Approximations I develop: more like gamma (right).
- (FYI: Data looks more like plot on the right.)

Edgeworth Expansion of Gamma

- What do expansions look like? If Y is average default time,

$$\hat{f}_Y(y) = \overbrace{\gamma_{\hat{\ell},\hat{\lambda}}(y)}^{\substack{\text{gamma}\\ \text{base}}} + \overbrace{\frac{\kappa_3\hat{\lambda}^3 - 2\hat{\ell}}{6}\sum_{i=0}^{3}(-1)^{3-i}\binom{3}{i}\gamma_{\hat{\ell}-i,\hat{\lambda}}(y)}^{\text{skewness correction}}$$
$$+ \overbrace{\frac{\kappa_4\hat{\lambda}^4 - 6\hat{\ell}}{24}\sum_{i=0}^{4}(-1)^{4-i}\binom{4}{i}\gamma_{\hat{\ell}-i,\hat{\lambda}}(y)}^{\text{kurtosis corrections}} \qquad (4)$$
$$+ \frac{(\kappa_3\hat{\lambda}^3 - 2\hat{\ell})^2}{72}\sum_{i=0}^{6}\binom{6}{i}(-1)^{6-i}\gamma_{\hat{\ell}-i,\hat{\lambda}}(y)$$
$$+ O(n^{-3/2})$$

- Mean, variance, skewness, kurtosis: $\frac{\hat{\ell}}{\hat{\lambda}}, \frac{\hat{\ell}}{\hat{\lambda}^2}, \kappa_3, \kappa_4$.

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Economic Meaning of Parameters

- Edgeworth expansion parameters yield economic insight.
  - Imply approximating portfolio of iid loans.
- $\hat{\ell} =$ iid-equivalent loan count (diversity score).
  - Unrelated, equal-size loans needed for similar default risk.
  - "This portfolio defaults like a portfolio of $\hat{\ell}$ iid bonds."
  - Thus $\hat{\ell}$ measures portfolio default-relative diversification.
- $\hat{\lambda} =$ iid-equivalent default rate.
  - Measures credit quality/default probability of iid loans.

Advantages of This Approach

Advantages over prior work:

- Theoretically-based *vs.* Schorin and Weinreich (1998).
  - SW: commonly used with Moody's KMV; criticized as *ad hoc*.
- ℓ, λ joint estimation *vs.* Duffie and Gârleanu (2001) ℓ.
  - DG: Only diversity score, no credit quality: admitted weakness.
- Pseudocumulants $\left(\frac{\kappa_3\hat{\lambda}^3-2\hat{\ell}}{6}, \frac{\kappa_4\hat{\lambda}^4-6\hat{\ell}}{24}\right) \Rightarrow$ default clusters/heavy tails.
  - Handles tail risk discussed in Duffie and Gârleanu (2001).
- May be used for forecasting default correlations.

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Caveats for This Approach

- One problem: We have switched terminology.
- Want: portfolio default times; instead model *average* default times.
- Often use Edgeworth expansions to model non-average distributions.
- Chambers (1967): this is OK if regularity conditions hold.
- Further, the ideas may still translate to the portfolio.
- Caveats:
  - Expansions may yield areas of “negative probability.”
  - Portfolio cumulants estimated via censored individual loans.

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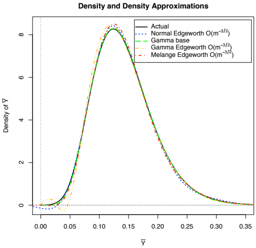
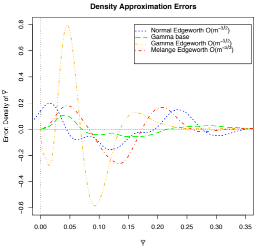
Simulation: 200-bond CDO Equity Tranche

- Simulate 5% equity tranche of a 200-bond subprime CDO[4].
- Risk factor: US economy. (average 1 event/20 years)
- Mean unaccelerated default times 5–20 years (BBB–BB credit).
- Crisis-accelerated default times 1–4 years (B–CCC credit).
- Use cumulants of equity tranche (first 10) default times.





MSEs:

Normal Edge.: 0.0034

Gamma base: 0.0006

Gamma Edge.: 0.0306

Mèlange Edge.: 0.0051

[4] Exaggerated number of bonds (200 > 125) for illustration.

Simulation: Interpretation

- Can see value of approximating portfolio (gamma base).
- 10 bonds in equity tranche: diversity score $\hat{\ell} = 7.8$ bonds.
- The 7.8 iid bonds would have default rate $\hat{\lambda}$ =6/year.
- Thus mean time to default $\approx$ 2 months (C credit).
- Tranche distribution implied by approximating portfolio:
  - Tranche life $\sim$ *Exp*$(\hat{\lambda}/\hat{\ell})$.
  - Mean tranche life: 15.6 months.

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Estimating the Approximating Portfolio

- All this theory begs the question: How do we use this?
- Let's consider a portfolio of 25 subprime (C-credit) loans.
- Walk through example estimation of approximating portfolio.
- *N.B.* doing this *a priori* is inherently forecasting.
- Will need a few pieces of data:
  - Occurrences of a systematic event (*e.g.* NBER recessions);
  - Old same-credit loans bridging systematic risk event.
- *N.B.* Use physical default rates to get at idiosyncratic rates.
  - CDS's mix systematic, idiosyncratic rates; hard to handle.

Estimating Default Acceleration, Idiosyncratic Credit

- Old loans $\Rightarrow$ MLE for default acceleration δ.
- Also $\Rightarrow$ coherent estimate of idiosyncratic identical-credit λ_i.

$$\mathcal{L}(\lambda,\delta|t_s) = \underbrace{\prod_{j\in\{\text{defaulted}\},t_j<t_s} \lambda_i e^{-\lambda_i t_j}\cdot(1-e^{-\lambda_s t_s})}_{\text{pre-crisis defaults}} \times \underbrace{\prod_{j\in\{\text{defaulted},t_j\geq t_s\}} \delta\lambda_i e^{-\delta\lambda_i(t_j-t_s)}\cdot e^{-\lambda_s t_s}}_{\text{in-crisis defaults}} \times \underbrace{\prod_{j\in\{\text{undefaulted,repaid}\}} e^{-\delta\lambda_i(T_j-t_s)}\cdot e^{-\lambda_s t_s}}_{\text{undefaulted (censored default)}}. \quad (5)$$

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Estimating Default Rate Parameters

- NBER: mean US business cycle of 55 months $\Rightarrow \lambda_s = 0.218$.
- 20 old loans (default times):

Pre-crash defaults	3.2	4.8	5.7					
Post-crash defaults	5.8	5.8	5.8	5.9	5.9	6.0	6.1	6.2
	6.3	6.5	6.8	7.2	7.7	8.3	9.2	
Repaid	10.0	10.0						

- MLE, in-crisis default acceleration $\hat{\delta} = 3.28$.
- MLE, idiosyncratic rate of default $\hat{\lambda}_i = 0.22$.

Forecasting Default Correlations

- Return to our 25 subprime loans.
- Simulate idiosyncratic defaults, systematic event times.
  - No closed-form solution; default acceleration is not affine.
- 10,000 simulations give these average default time cumulants:

$\hat{\kappa}_1$	$\hat{\kappa}_2$	$\hat{\lambda}$	$\widehat{\text{df}}$	$\hat{\ell}$
3.170	6.368	0.498	0.634	15.841

- Implies diversity score of $\hat{\ell} = 15.8$, 37% reduction.
- Approximating portfolio mean credit quality $\hat{\lambda} = 0.5$.
- Thus 25 C-credit loans which default at $3\times$ rate in recession...
- ...have default behavior like 16 D-credit loans.

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Conclusion

- Saw there are problems with affine models
- In-crisis default acceleration may break affine models.
  - Or may yield "time-varying" rates; thick tails.
- Found distribution approximation for mean bond default time.
  - Leads to an elegant (novel?) Edgeworth expansion.
  - Consistent for structural model of interacting "alarms."
- Approximating portfolio parameters also have economic meaning:
  - $\hat{\ell}$: diversity score = approximating iid loan count.
  - $\hat{\lambda}$: approximating iid loan default rate.
  - Jointly determined so as to be coherent.
- May be used to imply default distribution for tranche/portfolio.

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