A Network Model of Counterparty Risk

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Counterparty Risk

- *Counterparty*: other side of ongoing financial agreement.
  - A bank enters into a swap with you on the S&P 500.
- Counterparty Risk
  - Risk resulting from default/bankruptcy of a counterparty.
  - Strictly: Risk to you from one of your counterparties.
  - Broadly: Includes effects on overall market (our concern).

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Counterparty Risk: Why We Care

- Affects overall market when large bankruptcy looms/occurs:
  - Near-bankruptcy of Bear Stearns (May 2008)
  - Bankruptcy of Lehman Brothers (Sep 2008)
  - Bankruptcy of Refco Inc? (Oct 2005, owned #1 CME broker)
- Outstanding notional at CME before ceasing trading:

Bear	Lehman	Refco LLC
$761 BB	$1,150 BB	$130 BB

  - N.B. No defaults or trade halts at CME for these events.
- Other bankruptcies: Askin (1994), LTCM (1998, why I care)
  - Counterparty risk: concern. . . and "accelerant?"

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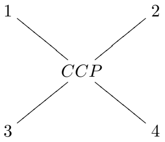
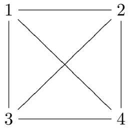
Model

Network Topologies

- Investigate two extremes of *n*-counterparty networks.



Star network
(Futures market w/CCP[1])



Fully-connected network
(Bilateral OTC market)

- Each node is a counterparty (capital K, risk aversion λ).
- Each edge is a contract[2] linking counterparties i and j
- Contract exposure: $q_{ij} = -q_{ji}$; $q_{i<j} \overset{iid}{\sim} N(0, \eta^2)$
- Counterparty i's net exposure: $Q_i = \sum_{j \neq i} q_{ij}$.
- Same net exposures (Q_i's) in both networks.

[1]Centralized counterparty.
[2]A swap or forward on a risky asset.

Event Timing

To study counterparty risk, events occur at discrete times.

$t = 0$: Bankruptcy of counterparty n occurs.

- All contracts with counterparty n are invalidated.
- Pushes unwanted exposure onto other $n - 1$ counterparties.

$t = 1$: Living counterparties trade in response to bankruptcy.

$t = 2$: Living counterparties close out bankruptcy-induced exposure.

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Price Impact of Trading

- Huberman and Stanzl (2004) arbitrage-free price impact.
  - Impact has linear permanent component.
  - Permanent component impacts prices for later traders.
- Each counterparty i trades x_i shares at time $t = 1$.
- Expected trade price for counterparty i at $t = 1$:

$$E(p_{i,1}) = p_0 + \underbrace{\pi x_i}_{\text{impact}} \tag{1}$$

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Price Evolution

- Trading occurs during periods 1 and 2:
  - The order of trading is random, not strategic; and,
  - Ordering and price impact create low and high prices.
- Time periods are very short; two simplifying assumptions:
  1. Prices have no drift other than price impact due to trading.
  2. Price diffusion is Gaussian (not log-normal).
- Thus the price at the end of period 1 is:

$$p_1 = p_0 + \sigma Z_1 + \pi \sum_{j=1}^{n-1} x_j \tag{2}$$

where $Z_{t \in \{1,2\}} \overset{iid}{\sim} N(0,1)$.

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Effects of Invalidated Contracts

- Bankruptcy invalidates each contract with exposure q_{in}.
- Star network: only contract with CCP is invalidated.
- Fully-connected network:
  - Each counterparty has unwanted exposure of $-q_{in}$
  - Net unwanted exposure: $\sum_{i \neq n}(-q_{in}) = \sum_{i \neq n} q_{ni} = Q_n$.
- Full rehedge (in either network) implies net trade of $-Q_n$.
- However, counterparties trade in own interest.
  - Do they rehedge immediately? Push market further?

Small Bankruptcy

Small Bankruptcy

- First consider bankruptcy of a small financial firm.
- Cause of bankruptcy may be market factors or idiosyncratic.
- What do we know about net exposure to the bankrupted?
  - Net exposure is likely to be small;
  - Possible non-market causes; cannot estimate net exposure.
- Each counterparty maximizes mean-variance utility:

$$U_i(x) = \underbrace{-\pi x^2}_{\text{period 1 impact}} \underbrace{-\lambda \frac{\sigma^2}{2}[q_{in}^2 - (x - q_{in})^2]}_{\text{variance penalty}} \underbrace{-\pi q_{in}(x - q_{in})}_{\text{period 2 impact}} \tag{3}$$

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Small Bankruptcy: Optimal Trade

- The optimal trade size is then given by:

$$x_i = \frac{(\pi + \lambda\sigma^2)q_{in}}{2\pi + \lambda\sigma^2}. \tag{4}$$

- Higher impact splits trades: $\pi \uparrow \infty \Rightarrow x \rightarrow q_{in}/2$; and,
- Higher volatility, hedge faster: $\sigma \uparrow \infty \Rightarrow x \rightarrow q_{in}$.

Small Bankruptcy: Added Volatility

- How much volatility does this trading add?
- Recall that $q_{i<j} \overset{iid}{\sim} N(0, \eta^2)$.
- Variance added to prices in period 1 due to exposures q_{in}:

$$\mathrm{Var}(p_1) = \sigma^2 + \underbrace{\pi^2(n-1)\left(\frac{\pi + \lambda\sigma^2}{2\pi + \lambda\sigma^2}\right)^2 \eta^2}_{\text{added variance}}. \qquad (5)$$

- This result applies only to fully-connected network.
- Ignore variance in period 2; may have setup-related artifacts.

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Large Bankruptcies

Large Bankruptcy

- Next consider the bankruptcy of a large financial firm.
- Assume large market move r_0 at $t = 0$ induces bankruptcy.
- Net exposure likely to be large; estimate Q_n via EVT.

$$\hat{Q}_n = \frac{-K}{r_0} + \frac{\eta\sqrt{n-1}}{c_n(1 - e^{-e^{-c_n\kappa_1 - d_n}})} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{-k(c_n\kappa_1 + d_n)}}{kk!} \tag{6}$$

where $\kappa_1 = \frac{-K}{r_0\eta\sqrt{n-1}}$, $c_n = \frac{1}{\sqrt{2\log(n)}}$, and
$d_n = \sqrt{2\log(n)} - \frac{\log\log(n) + \log(16\tan^{-1}(1))}{2\sqrt{2\log(n)}}$.

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Large Bankruptcies

- For large Q_n, trading at $t = 1, 2$ will move market a lot.
- Move will be further in direction that caused bankruptcy.
- This raises two distressing possibilities:
  - Move might greatly weaken other counterparties; or even,
  - A counterparty's hedging might bankrupt itself[3].
- Counterparties anticipate this, respond selfishly.
- Thus network structure matters.

[3] Checkmate.

Network Differences

- For a star network, only the central counterparty trades.
  - Eliminates expectations of net exposure, trading.
  - Matches real world: CCP can penalize predatory traders.
  - However, CCP must still worry about follow-on bankruptcies.
  - Optimization yields fraction $\gamma \in [0, 1]$ traded in $t = 1$.
- For fully-connected network, all counterparties may trade.
  - All estimate net exposure $\hat{Q}_n$ to be rehedged.
  - All anticipate follow-on bankruptcies to hedge $\hat{Q}_f$.
  - Trouble arises: $\gamma > 1$ to be expected.
  - Longs, shorts may largely self-segregate rehedge timing.

Large Bankruptcy: Equilibrium CCP Trade

- Why not proceed as before?
- CCP must anticipate follow-on bankruptcies.
- Equilibrium involves market impact, follow-on exposure $\hat{Q}_f$:

$$\kappa_2 = \frac{-Kp_0/[\eta\sqrt{n-1}]}{p_0 r_0 - \pi(\hat{Q}_n + \hat{Q}_f)}, \tag{7}$$

$$\hat{Q}_f = (n-1)^{3/2}\eta\frac{\phi(\kappa_2) - \phi(\kappa_1)}{\Phi(\kappa_1)}. \tag{8}$$

- Also interesting: # follow-on bankruptcies $\hat{b}$:

$$\hat{b} = (n-1)\frac{\int_{\kappa_2}^{\kappa_1}\phi(z)dz}{\int_{-\infty}^{\kappa_1}\phi(z)dz} = (n-1)\left(1 - \frac{\Phi(\kappa_2)}{\Phi(\kappa_1)}\right) \tag{9}$$

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Large Bankruptcy: OTC Trading Creates Highs, Lows

- OTC traders anticipate one another, follow-on bankruptcies.
- However: those most at-risk rehedge quickly, others delay.
- Random trade sequence yields uncertain rehedge path S_{n-1}.
- Low is important; affects extent of follow-on bankruptcies.
- Can estimate low $\underline{S_{n-1}}$ with a Brownian bridge:

$$E(\underline{S_{n-1}}) = -\gamma(\hat{Q}_n + \hat{Q}_f) - 4\tan^{-1}(1)\gamma\eta\sqrt{n-1}\cdot \phi\left(\frac{\gamma(\hat{Q}_n + \hat{Q}_f)}{\eta\sqrt{n-1}}\right)\left(1 - \Phi\left(\frac{\gamma(\hat{Q}_n + \hat{Q}_f)}{\eta\sqrt{n-1}}\right)\right). \quad (10)$$

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Large Bankruptcy: Equilibrium OTC Net Trade

- Then use this to solve for equilibrium OTC net trade.

$$\kappa_2 = \frac{-Kp_0}{\eta\sqrt{n-1}(p_0 r_0 + \pi E(\underline{S_{n-1}}))}, \tag{11}$$

$$\hat{Q}_f = (n-1)^{3/2}\eta\frac{\phi(\kappa_2)-\phi(\kappa_1)}{\Phi(\kappa_1)}. \tag{12}$$

- Important to note that $\gamma \geq 1$ (in $E(\underline{S_{n-1}})$).
- Finding γ is hard: n-player (random) game.

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Utility Functions: Player i

- Finding γ requires each player i's utility function:

$$\hat{U}_i(x_i; y_i := \sum_{j \neq i} x_j) =$$
$$\underbrace{-\lambda \frac{\sigma^2}{2} \left[q_{in}^2 + \left(\frac{\hat{Q}_f}{n - \hat{b} - 1} - q_{in} + x_i \right)^2 \right]}_{\text{variance penalty}} \underbrace{-\pi \left(\frac{\hat{y}_i}{2} + x_i \right) x_i}_{\text{period 1 impact}}$$
$$\underbrace{-\frac{\pi}{2} \left(q_{in} + \hat{y}_i - \hat{Q}_n - \frac{\hat{Q}_f (n - \hat{b})}{n - \hat{b} - 1} \right) \left(\frac{\hat{Q}_f}{n - \hat{b} - 1} - q_{in} + x_i \right)}_{\text{period 2 impact}} \tag{13}$$

- Simulations thus far: $\gamma > 1$. (1.5, 2?)

Checkmate

Proposition (Checkmate)

In a fully-connected network, there is a $Q_n \in (0, \infty)$ *such that for some* $k < n$ *and any finite* x_k *we expect bankruptcy in period 1:* $E(\pi \frac{Q_k}{p_0} \sum_{j<n} x_j | \mathcal{F}_1) > K - Q_k r_0$.

Proposition 1 means a large enough initial bankruptcy may result in an expected follow-on bankruptcy despite the best efforts of the checkmated counterparty.

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Hunting

Proposition (Hunting)

In a fully-connected network of 3 or more counterparties, there is a $Q_n \in (0, \infty)$ *such that for all exposures of* Q_n *or greater, bankruptcy has a positive expected payoff for two or more other counterparties.*

A sketch of the proof for $n = 3$ offers insight into hunting.

Proof.

Assume counterparty 3 is checkmated. Let $Q_1, Q_2 < 0 < Q_3$ be such that $Q_1 + Q_2 = -Q_3$. Wlog, assume $q_{13} = Q_1$. For $\pi > 0$, counterparties 1 and 2 trade Q_1 and Q_2 and can expect to bankrupt counterparty 3: $\pi(Q_1 + Q_2)Q_3 + K < 0$. Counterparties 1 and 2 finish with original exposures and MTM cash. □

Examples

Small Bankruptcy: Results

- Use sensible parameters[4] and $n = 10$ counterparties:

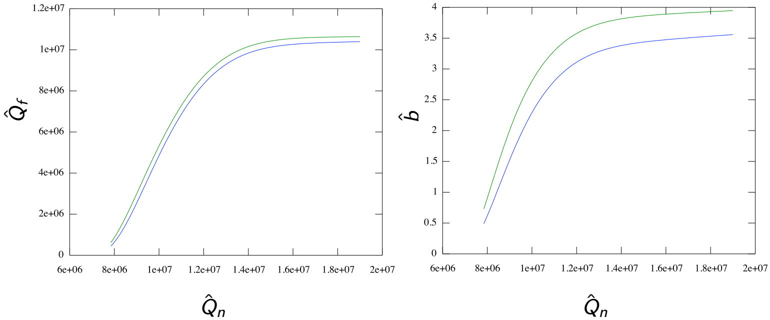
$$p_0 = \$50.00 \qquad \sigma = \$0.95 \text{ (30\% annual)}$$
$$\lambda = 1 \times 10^{-6} \qquad \eta = 100,000$$
$$\pi 2 \times 10^{-6} \qquad \text{volume} = 5 \text{ MM shares/day}$$

- Period 1 price impact: \$0.20.
- Period 1 volatility: $\$1.30 = 1.37 \times \$0.95$
- On an annualized basis, volatility went from 30% to 41%.
- In this model, higher volatility only lasts two periods.

[4] Impact parameters are as derived in Almgren and Chriss (2001).

Large Bankruptcies: Indicative Distress

- Consider large bankruptcy for $n = 10$ counterparties.
- Same parameters (except $\eta = 1,000,000$, $\gamma = 1.75$).



- Distress exposure $\hat{Q}_f$ and pervasiveness $\hat{b}$ vs. $\hat{Q}_n$.
- Top lines are for OTC market; bottom lines for CCP market.

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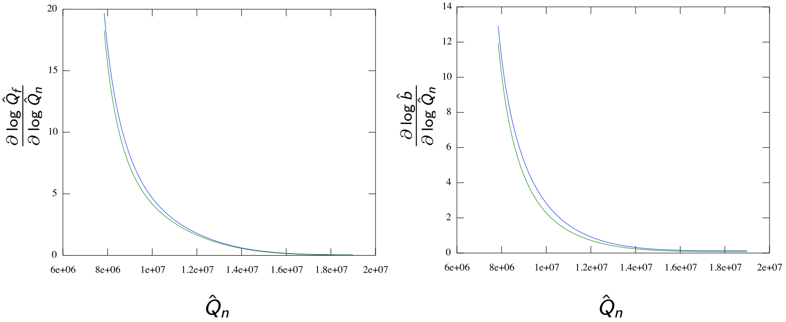
Large Bankruptcies: Indicative Elasticities

- Also look at elasticities of distress (exposure, pervasiveness).



- Elasticities of distress exposure $\hat{Q}_f$, pervasiveness $\hat{b}$ vs. $\hat{Q}_n$.
- Top lines are for OTC market; bottom lines for CCP market.

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Large Bankruptcies: Example of Market Impact

- Suppose $\hat{Q}_n = 3{,}000{,}000$.
- Assume fully-connected network hunts, trades $2\hat{Q}_n$ at $t = 1$.
    - Expected market impact: \$12.00.
    - Period 1 volatility: $\$17.83 = 18.77 \times \$0.95$
    - On an annualized basis, volatility went from 30% to 563%.
- Preliminary findings: This example may be conservative. (!)

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Large Bankruptcies: Not So Random

- Fully-connected networks admit two destabilizing events:
  - Checkmate: weak counterparty may have no beneficial trade.
  - Hunting: counterparties force others into bankruptcy.
- Worse, hunting is a full equilibrium behavior.
  - Market may be pushed far beyond one follow-on bankruptcy.
- Are counterparties selfishly amoral/evil? Maybe not.
  - Trade amount may pre-hedge expected follow-on bankruptcies.
  - This reduces surprise need for trading in period 2.
- Star networks have fewer such destabilizing events.
  - Suggests central clearing reduces OTC market volatility.

Remaining Work

- Still not sure I've thought through all effects.
- Odd: volatile rehedging $\Rightarrow$ low price effect is small.
- Find formula/approximation for γ based on exposures?
  - Might require solving the n-player game.
- Handle non-closed nature of trading?
- Extend to case of multiple dealers/"CCPs".

Conclusion

From a simple OTC market with price impact, we've seen that:

- Even small bankruptcies temporarily increase volatility.
- Large bankruptcy effects depend on network structure.
- For a large bankruptcy in a fully-connected network:
    - Counterparties may be unable to save themselves (checkmate).
    - Counterparties may hunt their weakest peers for profit.
- A large bankruptcy in a CCP network induces less distress.
- Suggests benefits to centralized clearing in OTC markets.
- Use model to estimate volatility externality cost.
    - Might suggest when to move products to central clearing.
- Measure when markets are more/less brittle?
- Sufficient info to trade leverage "emissions" credits?